

Ensemble Renormalization Group for Disordered Systems

Maria Chiara Angelini,¹ Giorgio Parisi,² and Federico Ricci-Tersenghi²

¹*Dipartimento di Fisica, Università “La Sapienza”, P.le A. Moro 2, I-00185 Roma, Italy*

²*Dipartimento di Fisica, INFN – Sezione di Roma1, CNR – IPCF UOS Roma, Università “La Sapienza”, P.le A. Moro 2, I-00185 Roma, Italy*

We study a Renormalization Group transformation that can be used also for models with quenched disorder, like spin glasses. The method is based on a mapping between disorder distributions, chosen such as to keep some physical properties (e.g. the ratio of correlations) invariant under the transformation. We validate this Ensemble Renormalization Group by applying it to the hierarchical model (both the diluted ferromagnetic version and the spin glass version), finding results in agreement with Monte Carlo simulations.

The renormalization group (RG) is a fundamental tool in theoretical physics [1]. It allows to characterize phase transitions and critical phenomena, by computing critical exponents and universality classes. The real space RG can be viewed as a decimation procedure that takes a system made of N dynamical variables and reduces it to a smaller system, in a way which preserves, or scales appropriately, some important physical observables. Such a decimation induces a RG transformation on the system couplings, and the study of such a transformation allows one to identify critical points and critical exponents.

Real space RG transformations have been studied in great detail for homogeneous models [1], but much less is known for disordered models, that contain quenched randomness in the Hamiltonian (either random fields and/or random couplings). This is specially true for frustrated models, like spin glasses (SG), for which a satisfying RG transformation is still lacking. For example, for the Edwards-Anderson SG model [2] on a D -dimensional lattice all the attempts to develop a field theory by performing an ϵ -expansion around the upper critical dimension $D_u = 6$ have proved to be very complicated [3, 4]. These studies have led to the discovery of fixed points different from the mean-field (MF) ones, however the implications of that are not completely clear. In particular the existence of replica symmetry breaking (RSB) fixed points in the non-mean field region $D < D_u$ has been shown only very recently [5] and estimates of critical exponents in $D = 3$ are still not reliable.

In this framework the development of a (semi-)analytical real space RG for disordered models would be very welcome. The outcome of such a RG transformation could be well compared with Monte Carlo (MC) simulations, that provide accurate estimates of critical temperatures and critical exponents for disordered models. Previous attempts of developing such a real space RG for disordered systems [6] focused on transformations mapping a single sample of size N to a smaller system (without loss of generality we can set the size of the smaller system to $N/2$). In formulae, we can write the mapping $\{J_{ij}\} \rightarrow \{J'_{ij}\}$ as the one solving a set of equations like

$$\langle O_k(\{J_{ij}\}) \rangle = \langle O'_k(\{J'_{ij}\}) \rangle, \quad (1)$$

where angular brackets are thermal averages with respect to the Gibbs-Boltzmann distribution, primed quantities refer to

the smaller system and the number of observables O_k are enough to determine the new couplings $\{J'_{ij}\}$. Given an ensemble of systems of size N , the above transformation can be applied to each of them in order to obtain an ensemble of systems of size $N/2$. However we believe that such a mapping is suboptimal for models with quenched disorder and a better RG transformation should consider explicitly the average over the quenched disorder (as was done in Ref. [7]). What we are proposing is a mapping between probability distributions of couplings $P(J_{ij}) \rightarrow P'(J'_{ij})$ such that the following equations hold

$$\overline{\langle O_k(\{J_{ij}\}) \rangle} = \overline{\langle O'_k(\{J'_{ij}\}) \rangle}. \quad (2)$$

The overbar represents the average over the quenched disorder (i.e. the couplings in the present case). The rationale beyond this choice is that in models with strong disorder (like SG) sample-to-sample fluctuations may dominate thermal ones.

Two simple examples may help elucidating the limits of the RG transformation working sample by sample, Eq.(1), and thus justify the use of the one in Eq.(2), that we will call Ensemble RG (ERG). In a diluted ferromagnet, where couplings are positive with probability p and null with probability $1 - p$, a single step of the decimation procedure induced by Eq.(1) typically generates all non-zero couplings (i.e. $p = 1$). And this is clearly not very useful if one is willing to follow the RG flow in the $p - T$ plane. Moreover, in frustrated models the decimated system is typically much less frustrated than the original one: the extreme case is the transformation of a 4-spins system in a 2-spins system, being the latter unfrustrated for any coupling choice! This tendency to reduce frustration makes the RG using Eq.(1) clearly unfit to describe SG fixed points.

In principle, our ERG scheme can be applied to any disordered system. We choose here to apply it to the hierarchical model (HM), which is a particular one-dimensional long range model, whose Hamiltonian for $N = 2^n$ spins can be constructed iteratively in the following way [8]:

$$H_n(s_1, \dots, s_{2^n}) = H_{n-1}(s_1, \dots, s_{2^{n-1}}) + H_{n-1}(s_{2^{n-1}+1}, \dots, s_{2^n}) + c^n \sum_{i < j=1}^{2^n} J_{ij} s_i s_j. \quad (3)$$

In practice H_n is the sum of interactions at n different levels. We have studied three versions of this model: the ferromagnet (FM), where $J_{ij} = 1$; the diluted ferromagnet (DFM), where a random fraction $1 - p$ of FM couplings are set to zero; and the SG version [9], with Gaussian couplings $P(J) \propto e^{-J^2/2}$.

There are many reasons to test a new RG transformation on the HM. First, by properly tuning the topological factor c that controls how fast the couplings intensity decays with distance, the HM can emulate a D -dimensional short range (SR) model: $c \simeq 2^{-1-\frac{2}{D}}$ for DFM and $c \simeq 2^{(-1-\frac{2}{D})/2}$ for SG. These relations are exact around the upper critical dimensions because the long range HM and the SR D -dimensional model have the same field theory at leading order. In order to have a phase transition at a finite temperature, the c parameter must satisfy $\frac{1}{4} < c < \frac{1}{2}$ for the DFM and $\frac{1}{2} < c < \frac{1}{\sqrt{2}}$ for the SG. Lower bound values of c correspond to lower critical dimensions (and thus $T_c = 0$), while if c reaches the upper bound the energy is no longer extensive (thus $T_c = \infty$). For $c > 2^{-\frac{3}{2}}$ in DFM and $c > 2^{-\frac{3}{2}}$ in SG, the model shows mean field critical properties (like for $D > D_u$ in SR models). So, tuning a single parameter in the HM, we can move from the MF region to a non-MF one.

The second reason to choose the HM, is that if the system is decimated by a standard block-spin transformation, the new Hamiltonian does not contain any multi-spin term (at variance to what happen on finite dimensional lattices [7]). So, considering only pairwise interactions in the RG is not an approximation for the HM. Moreover the FM version can be exactly solved in a time growing only polynomially with N , since the probability distribution of the magnetization satisfies

$$p_n(m) \propto e^{\beta c^n m^2} \sum_{m_L, m_R} p_{n-1}(m_L) p_{n-1}(m_R) \delta_{m_L+m_R, m}$$

where m_L and m_R are the magnetizations of the half systems.

We describe now in detail how to apply the ERG to the HM. We assume couplings to remain independent during the RG, but we allow couplings to have a different probability distribution at each level: in the original HM all couplings have the same probability law, but we have seen that the RG iteration produces different couplings at different levels. Each coupling distribution is parametrized by $K \in \{1, 2\}$ few numbers (that is the mean for FM, the variance for SG and the fraction of non-zero couplings and the mean for DFM), otherwise the search for a solution to Eq.(2) would become too difficult. We start from a system with n levels that we want to reduce to an “equivalent” smaller system of $n - 1$ levels.

1. First we compute $(n - 1)K$ observables $\overline{\langle O_k \rangle}$ in the larger system (in the FM the overbar can be omitted).
2. Then we identify the distributions of couplings in the smaller system by requiring that $\overline{\langle O_k \rangle} = \langle O'_k \rangle$ for any $k \in \{1, 2, \dots, (n - 1)K\}$.
3. Finally we join two smaller systems with random couplings extracted from the original distribution, to obtain again a system of the original size.

The first two steps are the true renormalization steps, while the latter is required to obtain a final system size, that will allow

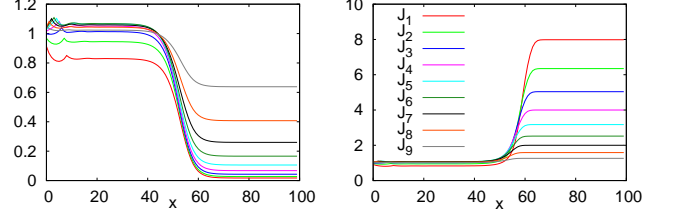


FIG. 1: (color online) Renormalized couplings J' in a FM system with $n = 10$ levels versus the number of RG steps, for temperatures slightly bigger (left) and smaller (right) than T_c .

us to iterate the method, until convergence. Thermal averages are computed exactly: this is easy to do in the FM, while in the DFM and in the SG we do it by exhaustive enumeration, thus limiting us to a small number of levels in the disordered cases. The average over the disorder is not exact, but taken over $\sim 10^5$ samples. Step 2 is actually accomplished by minimizing $\sum_k (\langle O_k \rangle - \langle O'_k \rangle)^2$, and we have checked that the reached minimum is always very close to zero. Since couplings distributions are different at each level, we do not see any better option than extracting the new couplings in step 3 from the original distribution.

The FM version of the HM, that has been solved analytically, is a benchmark for our numerical implementation of the ERG. For $c = 2^{-\frac{5}{3}}$, that corresponds to $D = 3$, the critical temperature is $T_c = 0.848154717$ [10], and the critical exponents are $\eta = 0$, $\gamma = 1.299140730159(1)$ [11], leading to $\nu = 0.649570365$ using the scaling relations (while for a 3D Ising FM they are $\nu = 0.6301(4)$, $\eta = 0.0364(5)$ [12]).

The observables O_k that we use in the RG equations are the correlation of the magnetization at level $k + 1$, normalized by those at level k , with $k \in \{1, \dots, n - 1\}$:

$$\langle O_k \rangle = \frac{\langle m_{L_k} m_{R_k} \rangle}{\langle m_{L_k} m_{L_k} \rangle}.$$

The denominator is necessary to reduce finite size effects and to ensure that a solution to the RG equations exists (this is not true in general for other observables, like the magnetization).

Applying the previously described procedure, the flux of couplings and correlations can be followed, see Fig. 1. If $T \simeq T_c$, renormalized couplings stay for a while close to the critical fixed point (FP), and then go towards the high temperature (HT) FP if $T > T_c$ or the low temperature (LT) FP if $T < T_c$. We estimate the critical temperature as the temperature dividing the flows towards the two different FP. Please note that the HT and LT fixed points are not characterized by the usual $J = 0$ and $J = \infty$ coupling values: the reason is that in step 3 of our procedure we put new coupling of original intensity. Nonetheless couplings flows clearly differentiate HT and LT behaviors.

In order to extract critical exponents from the RG equations, we focus on the early regime, when the coupling flows leave

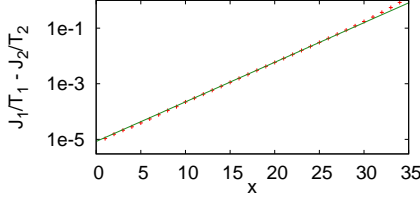


FIG. 2: Difference between the renormalized couplings at $T_1 = 0.84571$ and $T_2 = 0.845716$ in the FM HM with $n = 13$ levels and $D = 3$. The fit estimates the ν exponent.

the critical FP. From the Wilson relation an equation relating renormalized coupling after x RG steps can be obtained:

$$\frac{J_1(x)}{T_1} - \frac{J_2(x)}{T_2} = \left(\frac{1}{T_1} - \frac{1}{T_2} \right) b^{\frac{x}{\nu}},$$

where $b = 2^{\frac{1}{\nu}}$ is the scaling factor in our case. Thus, the ν exponent can be estimated from a fit like the one in Fig. 2. The values obtained for the critical temperature and the critical exponents in $D = 3$ are well comparable with the known ones: T_c extrapolates to $0.8478(1)$ in the large n limit and the measured ν exponent is $0.692(2)$ for $n = 13$ (although the extrapolation of ν to the $n \rightarrow \infty$ limit is much harder due to strong finite size effects). Moreover we have checked that our numerical RG recovers the right bounds on c , namely $T_c \rightarrow 0$ for $c = 1/4$ and $T_c \rightarrow \infty$ for $c = 1/2$.

We consider now the DFM. The Hamiltonian of the model is always the one in Eq. (3), but the couplings at level k are independent random variables extracted from the distribution

$$P_k(J) = p_k \delta(J - J_k) + (1 - p_k) \delta(J).$$

At the beginning $p_k = p$ and $J_k = 1$ for any k , while under the RG they will differentiate. The number of parameters to be determined in the ERG is $2(n - 1)$, and we use the following observables, with $k \in \{1, \dots, n - 1\}$, to fix them:

$$\overline{\left(\frac{\langle m_{L_k} m_{R_k} \rangle}{\langle m_{L_k} m_{L_k} \rangle} \right)}, \quad \overline{\left(\frac{\langle m_{L_k} m_{R_k} \rangle}{\langle m_{L_k} m_{L_k} \rangle} \right)^2}.$$

Applying the same procedure as for the pure model, we are able to draw a flow diagram in the $p - T$ plane for $D = 3$ and determine the critical line (see Fig. 3). The validity of the phase diagram found with the ERG is confirmed by a set of MC simulations [13] whose T_c estimates are also shown in Fig. 3. The only disappointment about this phase diagram is that we do not find an unstable FP along the critical line as expected for a $D = 3$ SR model [14]. However this can be explained by noticing that the α exponent of this model is very small, $\alpha = 0.051288905$, and so the crossover from the pure behavior can be extremely long.

Finally we study the SG version. The Hamiltonian is always the one in Eq. (3), and the couplings at level k are distributed with a Gaussian law of zero mean and variance σ_k^2

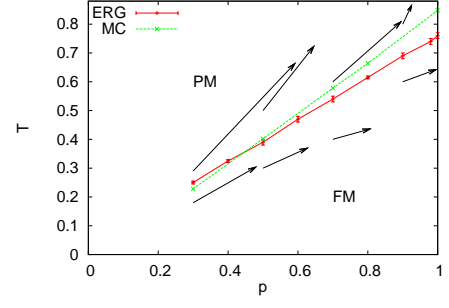


FIG. 3: Phase diagram in the $p - T$ plane for the 3D DFM as obtained by the ERG and by MC simulations. Arrows represent the first iteration of the ERG.

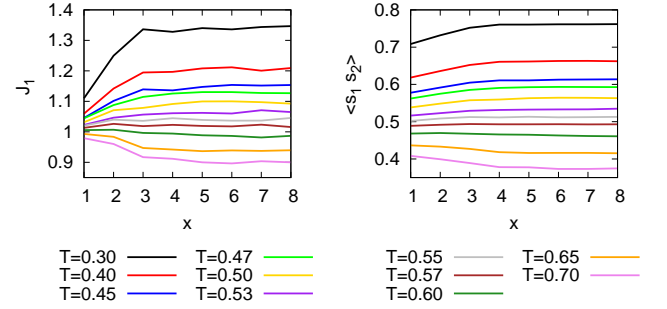


FIG. 4: (color online) Variances of renormalized couplings (left) and of SG correlations (right) at the lower level ($k = 1$) in a SG system with $n = 4$ levels and $D = 3$ versus the number of renormalization steps x , for many temperatures.

We locate the critical temperature at $T_{SG} = 0.58(1)$.

(at the beginning $\sigma_k^2 = 1$ for any k). The assumptions that the renormalized couplings are independent and normally distributed can be released by adding extra terms in the coupling distributions [7], but we leave these generalizations for future works. In the SG case the observables used to fix the $n - 1$ variances are SG correlations at different levels:

$$\overline{\langle O_k \rangle} = \frac{\sum_{i \in L_k, j \in R_k} \langle s_i s_j \rangle^2}{\sqrt{\sum_{i, j \in L_k} \langle s_i s_j \rangle^2 \sum_{i, j \in R_k} \langle s_i s_j \rangle^2}}.$$

Because of the computational costs we use $n \leq 4$, so the early regime leaving the critical FP is rather short, and the stationary regime is soon reached (with respect to the FM case). This effect is also enhanced by the disorder: indeed, even exactly at criticality, the SG ensemble contains many samples which are not critical, and the couplings of these samples flow away from critical values very fast. So, it seems unavoidable that disorder increases the instability of critical FP and consequently the uncertainty on the estimates of critical exponents. Nonetheless we can distinguish two temperature regions separated by a critical temperature T_{SG} (see Fig. 4 for an effective dimension $D = 3$ and $T_{SG} = 0.58(1)$), such that above

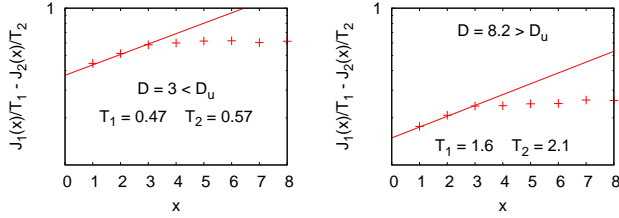


FIG. 5: Difference between the renormalization flux of the couplings at two different temperatures in a semi-log scale for $D = 3 < D_u$ (left) and $D = 8.2 > D_u$ (right). The first part, not affected by finite size effects, has been used to extract the ν exponent, through a power law fit.

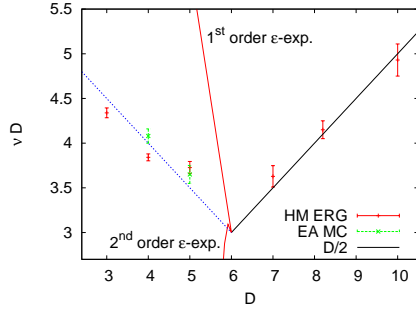


FIG. 6: The estimation of the ν exponent from the ERG at different effective dimensions D . The line for $D \geq 6$ is the MF prediction, while the one for $D < 6$ is just a guide for the eyes. Also results from the ϵ -expansion and ν values for the EA model in $D = 4, 5$ are added for comparison.

T_{SG} correlations and couplings decay towards zero, while below T_{SG} correlations and couplings variances grow, suggesting that the system is in a SG phase. In Fig. 4 we have plotted only couplings and correlations variances measured at the lowest level $k = 1$, but (as in the FM, see Fig. 1) the renormalized variances at the other levels are related to those at $k = 1$: for example, $\sigma_2 > \sigma_1$ if $T > T_{SG}$ and $\sigma_2 < \sigma_1$ if $T < T_{SG}$ (remember that parameters at the lowest level are those which have been renormalized the largest number of times).

Also in the SG case we are able to estimate the ν exponent from the flux of the couplings at early times. The procedure used is the same of that in the FM case and typical fits are shown in Fig. 5 for $D = 8.2$ in the mean-field region and for $D = 3$ below the upper critical dimension. We obtain $\nu D = 4.15(10)$ in $D = 8.2$ and $\nu D = 4.34(6)$ in $D = 3$. The mean field value is known analytically to be $\nu = 1/2$, in good agreement with our result. In order to check critical temperatures and the critical exponent in the non-mean-field region, we are running MC simulations with the parallel tempering algorithm (to be reported in detail in a future publication [13])

and preliminary data seem to scale rather well with critical temperatures and exponents obtained via the ERG method. In Fig. 6 we report the estimates of νD for several effective dimensions D : we see that in the mean-field region results are compatible with linear behavior $D/2$ and, more interestingly, the critical exponent νD have a minimum around the upper critical dimension $D_u = 6$. This minimum was not observed in previous RG studies [6], while it is present in SR models: in Fig. 6 we report the νD estimates for the EA model in $D = 4, 5$, which are close to the ERG estimates. The same cusp-like behavior for the νD exponent has been also seen in a 1D SG model with power law decaying interactions [15].

Concluding, we have developed a semi-analytical real space RG method that can be used for disordered systems. The method has been applied to the hierarchical model and is able to find a SG transition also for effective dimension $D = 3$ in the non-mean field region. The reliability of the method has been tested comparing the values of critical temperatures and critical exponents with those obtained in MC simulations and the agreement is satisfactory for all the versions we have studied: diluted ferromagnet and spin glass.

This work also solves an apparent inconsistency problem between the ϵ -expansion of the SG version [16] and another real space RG approach recently proposed in Ref. [6], where the νD exponent was found to decrease linearly with D , with no minimum at all around the upper critical dimension. In this work we have shown that such a minimum in νD exists if a better RG transformation is used.

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